Suppose (V_1, V_2) is a Heegaard splitting of M.

Let $\{D_1, \ldots, D_n\}$ denote a collection of properly embedded discs in V_2 which cut it into a ball.

We have $\partial D_i \subseteq \partial V_2 = \partial V_1$, and $\{\partial D_1, \ldots, \partial D_n\}$ cuts $\partial V_2 = \partial V_1$ into a 2-sphere with $2n$ holes.

The system $(V_1; \partial D_1, \ldots, \partial D_n)$ is

called a <u>Heegaard diagram</u> associated to the Heegaard splitting (V_1, V_2).

We can recover M from $(V_1; \partial_1, \dots, \partial_n)$; in fact, given a cube with n handles V_1 and a collection $\{\partial_1, \dots, \partial_n\}$ of pairwise disjoint circles in ∂V_1 which cut ∂V_1 into a 2-sphere with $2n$ holes, then $(V_1; \partial_1, \dots, \partial_n)$ is a Heegaard diagram of a manifold M

obtained as follows:

For every i, let $N(J_i)$ denote a n'hood of J_i in ∂V, homeomorphic to $S^1 \times [-1,1]$; pick $N(J_i)$'s so that they are pairwise disjoint.

Now we attach $B^2 \times I$ to V, along $\partial B \times I \xrightarrow{\cong} N(J_i)$, one for every i.

The outcome is a 3-mfld with boundary S^2. We fill this in using

a 3-ball.

By construction, $V_2 = \overline{M \setminus V_1}$ is a cube with handles, and so it is unique up to homeomorphism.

<u>Fundamental group:</u>

Let M admits a diagram $(V_1; f_1, \ldots, f_n)$.

Pick a free basis $x_1, \ldots, x_n$ of $\pi_1(V_1) \cong F_n$.

Now gluing $B^2 \times I$ to $N(f_i)$ has

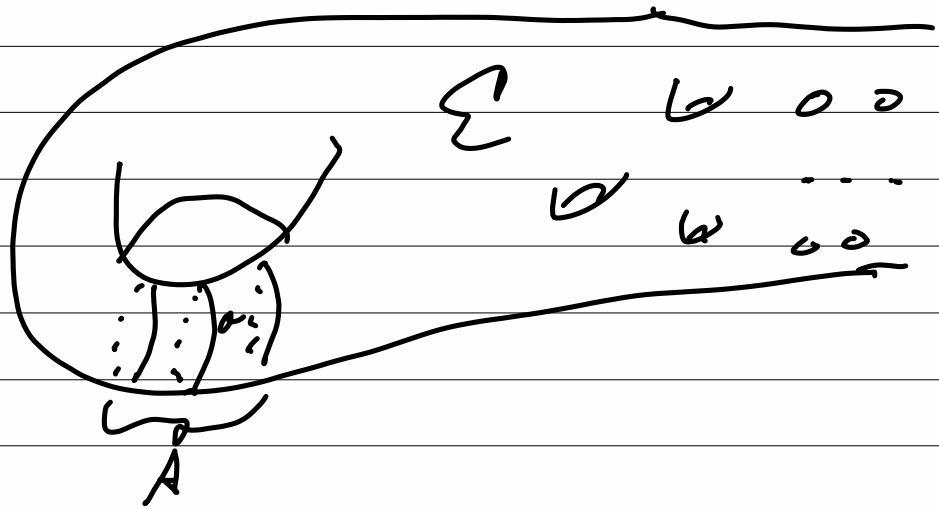
the effect of adding a relator r_{ij} adding the 3-ball at the end has no effect on π_1. Thus

$$\pi_1(M) = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle.$$

Note that Heegaard diagrams for a given Heegaard splitting are not unique.

We will need a tool from the theory of Mapping Class Groups (homeomorphisms of surfaces):

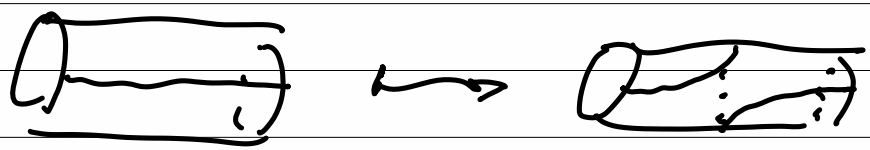
Def Σ a surface, σ a simple closed curve, A an annulus which is a regular n'hood of σ.



A **Dehn twist** around σ is a homeomorphism of Σ obtained by cutting along σ, then twisting A

$$A \cong \overset{\sigma}{S^1} \times I \longrightarrow S^1 \times I \cong A$$

$$(x, t) \longmapsto (x e^{2\pi i t}, t)$$

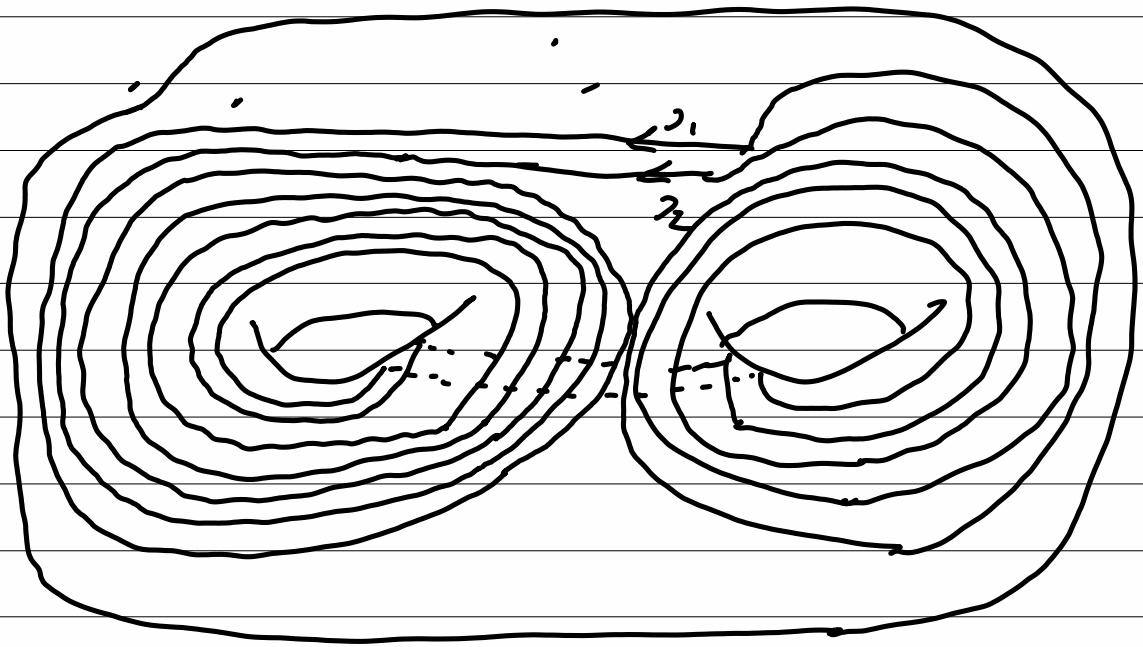


and then gluing A back to $\overline{\Sigma \setminus A}$.

Note that if we have a cube with handles M and $\sigma \subseteq \partial M$ bounds a disc, we can extend the Dehn twist from $\partial M \to \partial M$ to $M \to M$.

Exercise 2.6

Let $(V_1; \gamma_1, \gamma_2)$ be the following

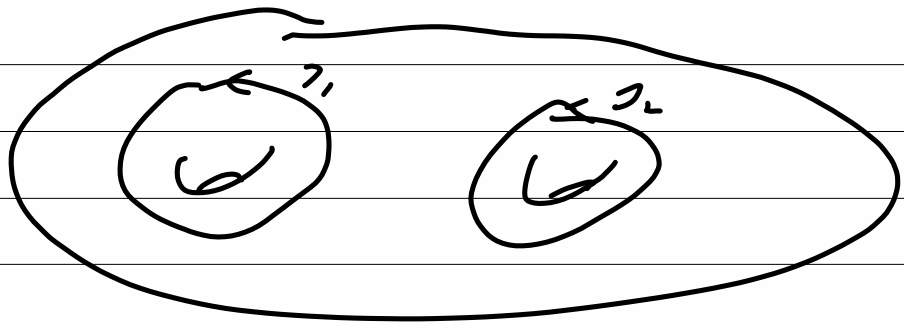


Heegaard diagram of a mfld M.

Note that

$$\pi_1(M) = \langle x_1, x_2 \mid x_1^2 x_2^3,\ x_1^3 x_2^4 \rangle \cong \mathbb{1}.$$

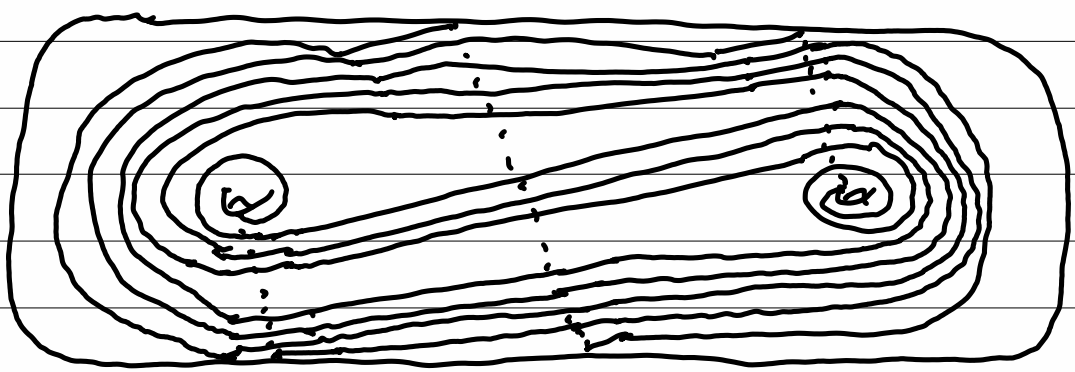
Show that M also admits the Heegaard diagram



and so that $M \cong S^3$.

<u>Example 2.7</u>

Consider the Heegaard diagram



We have

$$\pi_1(M) = \langle a, b \mid aba\bar{b}\bar{a}\bar{b},\ ab\bar{a}^{-2}ba\bar{b} \rangle$$
$$\overset{\shortparallel}{G}$$

Note that $G^{ab} = \langle a, b \mid [a,b], a\bar{b}, b \rangle = 1$.

So $H_1(M;\mathbb{Z}) = H_1(G;\mathbb{Z}) = G^{ab} = 0$.

Since M is connected, $H_0(M) = \mathbb{Z}$.

By Poincaré duality,

$$H_2(M) = 0, \quad H_3(M) = \mathbb{Z}.$$

Hence, M has the same homology as S^3. But:

$$G \longrightarrow S_5 = \mathrm{Sym}(5)$$

$$a \longmapsto (12345)$$

$$b \longmapsto (15324)$$

we verify $ab\bar{a}\bar{b}\bar{a}\bar{b}$ acts as:

$$1 \mapsto 4 \mapsto 3 \mapsto 5 \to 1 \mapsto 5 \to 1$$

2	3	2	3	4	1	2
3	5	4	2	3	2	3
4	1	5	4	5	3	4
5						5

and similarly $ab\bar{a}^{-2}ba\bar{b} \mapsto id$.

So we have constructed a non-trivial homomorphism $G \to S_5$, and so $G \neq 1$. Thus $M \not\cong S^3$.

[M is called a <u>homology sphere</u>].

<u>Genus one splittings</u>

Orientable case.

First consider genus 0 splitting: there is a unique diagram $(B^2; \emptyset)$, and the 3-manifold

decomposes as B^2 glued to B^2 along their boundaries.

This is clearly S^3.

Now consider genus 1 splittings.

We fix $S^1 = \{e^{2\pi i t} \mid t \in \mathbb{R}\} \subseteq \mathbb{C}$.

$T^2 = S^1 \times S^1 = \{(e^{2\pi i t}, e^{2\pi i s})\}$.

We label maps $S^1 \to T^2$ as follows:

$$a : t \mapsto (e^{2\pi i t}, 0)$$

$$b : t \mapsto (0, e^{2\pi i t}).$$

We will abuse notation by using

a and b to denote generators

of $\pi_1(T^2) = \langle a, b \mid [a, b] \rangle \cong \mathbb{Z}^2$.

Since $\mathbb{Z}^2 \cong \pi_1(T^2)$ is abelian,

every closed loop in T^2 determines

uniquely an element $a^p b^q \in \pi_1(T^2)$.